Question 1	Question 2	Question 3	Question 4	Sum	Final score

Written exam ('Quinto appello') of Teoria delle Funzioni 1 for Laurea Magistrale in Matematica - 26 September 2012.

SURNAME .... NAME .... MATRICOLA ....

PLEASE NOTE. During this exam, the use of notes, books, calculators, mobile phones and other electronic devices is strictly FORBIDDEN. Personal belongings (e.g., bags, coats etc.) have to be placed far from the seat: failure to do so will result in the annulment of the test. Students are entitled to use only a pen. The answers to the questions below have to be written in these pages. Drafts will NOT be considered. Marked tests will be handed out in room 1BC45 on 27 September 2012 at 14:00.

Duration: 150 minutes

Question 1.

(i) Let $N \in \mathbb{N}$, $N \geq 2$. Let $f \in C^1(\mathbb{R}^N \setminus \{0\})$. Assume that $f \in L^1_{loc}(\mathbb{R}^N)$ and the classical derivatives $\frac{\partial f}{\partial x_i}$ belong to $L^1_{loc}(\mathbb{R}^N)$ for all $i = 1, \dots, N$. Prove that the weak derivatives $(\frac{\partial f}{\partial x_i})_w$ of f exist in $\mathbb{R}^N$ and $(\frac{\partial f}{\partial x_i})_w = \frac{\partial f}{\partial x_i}$ almost everywhere in $\mathbb{R}^N$.

(ii) Is it true that the statement in (i) holds also for $N = 1$? Give a detailed motivation.

(iii) Is it true that if a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere in the classical sense then its weak derivative exists and coincides almost everywhere with its classical derivative?

Answer:

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Question 2.

(i) Give the definition of 'differential dimension' of a function space of the type $Z(\mathbb{R}^N)$.

(ii) Compute the differential dimension of the the space $w^{l,p}(\mathbb{R}^N)$.

(iii) Prove in detail (here any general result must be proved) that if the Sobolev space $w^{l,p}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ and $lp < N$ then $q \leq p^*$ where p^* is the usual Sobolev exponent associated with l, p, N.

Answer:

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Question 3.

(i) Let Ω be an open subset of $\mathbb{R}^N$. Give the definition of the Sobolev space $W_0^{l,p}(\Omega)$.
(ii) Prove that if a function $f \in W^{l,p}(\Omega)$ has compact support in Ω then $f \in W_0^{l,p}(\Omega)$.
(iii) Is it true that if $f \in W_0^{1,p}(\Omega)$ then $f \in L^\infty(\Omega)$? If yes, prove it. Otherwise, give a counterexample.

Answer:

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Question 4.

(i) State in detail the Sobolev's Representation Formula.
(ii) State in detail the Sobolev's Embedding Theorem part 1.
(iii) Give the definition of the Besov-Nikolskii spaces and state the Trace Theorem.